Proceedings of the 1997 IEEE
International Symposium on Assembly and Task Planning
Marina del Rey, CA - August 1997

A Computational Representation for Rigid and Articulated Assembly

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ABSTRACT

With the increasing level of automation in assembly planning and assembly execution, it becomes more obvious that there is a gap between the output of a mechanical designer and an assembly planner. The question is: How to describe a designed assembly to an assembly planning system? The input to almost all the current reported automatic assembly planning systems is one-static-state of the final assembly configuration regardless the assembly is meant to be rigid or articulated. The inability to represent the assembly design completely, accurately and computationally has hindered the power of an assembly planner in dealing with articulated assemblies as simple as taking something out of a drawer. In this paper we identify a computational representation (specification) of an assembly. The basic idea of this representation is to use each oriented *surface on a solid as a descriptive primitive and the* symmetry group *of the surface as a computational primitive. The relative motions (degree of freedom) under various contacts between a part or a subassembly and the rest of the assembly can be efficiently determined by computing these basic symmetry groups in a proved correct manner. The results reported in this paper lay out a more realistic and precise group theoretic framework than our previous work, and provide a concise, complete and computational representation for rigid and articulated assembly.*

1 Introduction

With the increasing level of automation in assembly planning and assembly execution, it becomes more obvious that there is a gap between the output of a mechanical designer and an assembly planner. The question is: How to describe a designed assembly to an assembly planning system? The input to almost all the current reported automatic assembly planning systems [14, 15, 9, 2] is one-static-state of the final assembly configuration regardless the assembly is meant to be rigid or articulated. The inability to represent the assembly design completely, accurately and computationally has hindered the power of an assembly planner in dealing with articulated assemblies as simple as taking something out of a drawer. By a *computational representation* of an assembly we mean that the representation of the assembly can be directly used to compute:

1. relative positions of its parts in the final assembly configuration
2. the type and range of motion of any subset of parts in the assembly
3. separation of subassemblies
4. compliant motion for (dis)assembly process

In this paper we identify a computational representation (specification) of an assembly that is composed of rigid solids[1]. The basic idea of this representation is to use each *oriented* surface on a solid as a descriptive primitive and the *symmetry group* of the surface as a computational primitive. The relative motions and the degrees of freedom under various contacts between a part or a subassembly and the rest of the assembly can be efficiently determined by computing these basic symmetry groups in a provably correct manner. Different from the study of solids in local contact such as [3, 10], our aim is to have a complete and precise description of the intended, possibly articulated, final assembly configuration where each part usually has multiple contacts with the rest of the assembly; and our approach is algebraic in nature. Different from our previous work [6] where the surfaces of a solid are treated as set points without taking orientations into consideration, in this work oriented surfaces are used as the basic building blocks. Also different from [11, 12, 13] in that a group theoretical formalism is embedded in a concise representation of an assembly such that no extensive algebraic equation manipulation is involved.

In Section 2 we establish the basic vocabulary — oriented surfaces and their symmetry groups — for describing assembly, and the relationship of a pair of oriented surfaces. Then in Section 3 we introduce compound feature as a result of considering multiple contacting surfaces between solids, and present results on

[1] A three dimensional continuum for which the distance between any pair of its points remains unchanged under any physically possible motion

computing the symmetry group of a set of oriented surfaces. In Section 4, examples are shown to illustrate the effectiveness of the approach. In Section 5 we summarize current results and discuss future work. All proofs of the results reported in this paper can be found in [8].

2 Basics: Oriented Primitive Feature and Its Symmetry Group

Since contacts among solids happen via the contacts of the surfaces of the solids, the representation and characterization of each contacting surface constitutes the foundation of any formalization for solid contacts. A group theoretic formalization of surface contact [6] treats each surface of a solid as (possibly infinite) a subset S in Euclidean space, which can be expressed as a polynomial. A symmetry of S is defined as:

Definition 2.0.1 *An isometry (a distance preserving mapping) g is a* **symmetry of a set** $S \subset \Re^3$ *if and only if* $g(S) = S$.

A single surface, treated as a set of points or with orientation vectors pointing inward, has the same symmetries as the surface with orientation vectors pointing outward[2]. However, in real world problems it is rare that only one surface is considered in isolation. In an assembly, it is often the case that multiple surfaces of one solid are in contact with multiple surfaces of other solids, this is a situation where a surface is treated as a set can run into problems: Figure 1

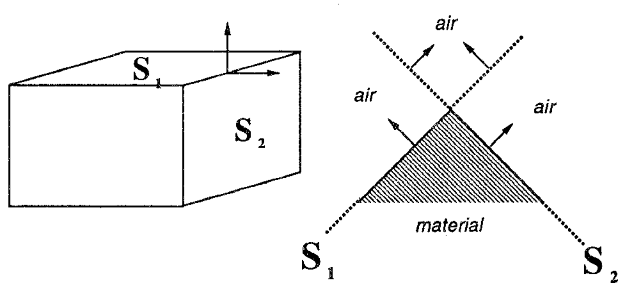

Figure 1: Two adjacent planes, S_1, S_2, on a cube

shows two adjacent (infinite) planar surfaces S_1, S_2 of a block. If the two surfaces are treated as sets the symmetries of the two planes include a 90^0 rotation about the line of the intersection of the two planes which is not a symmetry in reality. If one takes into consideration the fact that one side of the plane is the material of the solid and the other is the air, the only symmetries left are those 180^0 rotations that preserve both the bounding surfaces and their orientations. Another example of such non-real symmetries is illustrated in Figure 2. If the two cylindrical surfaces S_1, S_2 are treated as sets (infinite cylinders) then one cannot distinguish the two cases (a) and (b). In case (b) the cylindrical hole S_1 and the cylinder S_2, though they have the same radius, are not interchangeable if one takes their orientations into consideration.

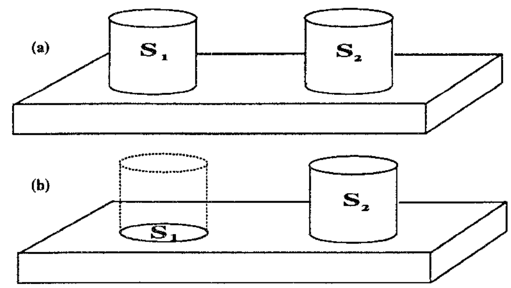

Figure 2: In both cases (a) and (b), S_1 and S_2 are interchangable if they are treated as sets.

Obtaining the accurate symmetry group of a set of contacting surfaces becomes crucial in applications where either an assembly planner needs to decide which assembly parts fit with each other [7, 4] based on whether they have compatible symmetry groups, or what are the relative motions between a subassembly and the rest of the assembly.

The aforementioned problems call for a more precise characterization of surface features of a solid, i.e. taking the orientations of a surface into consideration. This addition to a set-feature will require that the symmetries of a surface keep both the points on the surface and the orientations of the surface, respectively, setwise invariant. The group theoretical formalization, thus, needs to be re-evaluated given oriented surfaces as the descriptive primitives of a solid.

2.1 Oriented Surface and Its symmetry Group

We introduce the concept of *oriented features* by defining a set of outward-pointing normal vectors for each surface point of a solid. The polynomial used to express an algebraic surface implicitly, such as the ones provided by a geometric solid modeler, defines such normal vectors. Let $\mathcal{S}^2$ be the unit sphere at the origin embedded in $\Re^3$, each point of $\mathcal{S}^2$ corresponds to a unit vector in $\Re^3$.

Definition 2.1.1 *A solid M is a connected, rigid, three dimensional subset of Euclidean space $\Re^3$.*

Definition 2.1.2 *An* **oriented primitive feature** $F = (S, \rho)$ *of a solid M is an oriented surface where*

1) $S \subset \Re^3$ *is a connected, irreducible*[3] *and continuous algebraic surface which partially or completely coincides with one or more finite oriented faces of* M;

2) $\rho \subset S \times \mathcal{S}^2$ *is a continuous relation. For each* $s \in S$ *if s is a non-singular point of surface S (p.78 [1]) then* $v \in \mathcal{S}^2$ *is one of the two opposing*

[2]Planar surface is an exception: when it is treated as a set there are flipping symmetries which do not exist for oriented planes. In practice, this can be easily handled by checking the signs given by a solid modeler.

[3]Here *irreducible* implies that a primitive feature cannot be composed of any other *more basic* algebraic surfaces.

normals of the tangent plane at point s such that $(s,v) \in \rho$; if s is a singular point of S (e.g. at the apex of a cone) then, for all v, where $v \in \mathcal{S}^2$ is the limit of the orientations of its neighborhood, $(s,v) \in \rho$.

3) For all $s \in M, (s,v) \in \rho, v$ points away from M.

Intuitively speaking, a feature is composed of both "skin", S, and "hair", the set of normal vectors which correspond to the points on $\mathcal{S}^2$. Each element of relation ρ is a correspondence between a point on S and a vector on $\mathcal{S}^2$. Note, there may be more than one 'normal vector' at one point of a surface, e.g. at the apex of a conic shaped surface.

Let $\mathcal{E}^+$ be the proper Euclidean group which contains all the rotations and translations in $\Re^3$, $\mathbf{T}^3$ be the maximum translation subgroup of $\mathcal{E}^+$, and $SO(3)$ all the rotations about the origin. We now define how an isometry acts on the relation ρ defined in Definition 2.1.2:

Definition 2.1.3 *Any isometry $g = tr$ of $\mathcal{E}^+, t \in \mathbf{T}^3, r \in SO(3)$ acts on ρ in such a way that $(s,v) \in \rho \Leftrightarrow (gs, rv) \in g * \rho$.*

Now we define the symmetries for an oriented surface:

Definition 2.1.4 *A proper isometry $g \in \mathcal{E}^+$ is a* **proper symmetry of an oriented surface** *(primitive feature) $F = (S,\rho)$ if and only if $g(S) = S$* **and** *$g * \rho = \rho$.*

Note, the difference between the symmetries of a set (Definition 2.0.1) and this definition. There is an extra demand on a symmetry for an oriented surface — it has to preserve the orientations of the surface as well. Since orientations are points on $\mathcal{S}^2$, symmetries of an oriented feature have to keep **two** sets of points in $\Re^3$ setwise invariant. One can prove that the symmetries for an oriented surface form a group:

Proposition 2.1.5 *The symmetries of an oriented feature $F = (S,\rho)$ form a subgroup of $\mathcal{E}^+$, called the* **symmetry group of feature** *F.*

3 Multiple Contacts: Compound Features and their Symmetry Groups

An assembly is a manifestation of surface interactions of its subparts, albeit the physical property of each individual part (rigid or deformable) or the nature of the contact (static or articulated). Thus the representation of an assembly is reduced to how to specify a set of *contact constraints* which dictate the configuration of a set of solids. The key contribution of the group theoretical formalization is the transformation from a set of descriptive contact constraints to a set of computational constraints, i.e. the symmetry groups of multiple contacting surfaces. Our work has been carrying through the formalization, algorithm development and implementation stages. In this section we focus on setting up the framework for computing the symmetry group of multiple contacting surfaces.

From [4, 6] we have developed an expression for the relative motions of two solids B_1 and B_2 in contact via surfaces F_1 and F_2 respectively:

$$l_1^{-1} l_2 \in f_1 G_1 G_2 f_2^{-1}, \quad (1)$$

where $l_1^{-1} l_2$ is the relative position of solid 2 w.r.t. solid 1, G_1, G_2 are symmetry groups of F_1 and F_2 respectively, l_1, l_2 specify the locations of solids B_1, B_2 in the world coordinate system and f_1 and f_2 specify the locations of F_1, F_2 in their respective body coordinates. When two solids under n surface contacts, their contacting surfaces are coincide, thus the corresponding symmetry groups of the contacting surfaces are the same $G_1 = G_2 = G$, the relative positions:

$$l_1^{-1} l_2 \in f_1 G f_2^{-1}. \quad (2)$$

This expression has shown clearly that the possible motions of a solid or a subassembly S in an assembly can be described precisely by the symmetry group G of the multiple contacting oriented surfaces of S. If G is an identity group, i.e. $l_1^{-1} l_2 = f_1 f_2^{-1}$ gives a fixed position for S. If G is a finite rotation group, then $f_1 G f_2^{-1}$ contains a finite number of transformations reflecting the existence of multiple equivalent positions of subassembly S in the assembly. If G is a continuous group of dimension n then there exists relative continuous motions with d.o.f. n between S and the rest of the assembly. This is why explicitly expressing and effectively computing the symmetry group G of N surfaces are at the heart of the group theoretical formalization for solid contacts.

For more general cases:

1. Two solids have n general contact, the relative position of solid 2 with respect to solid 1:

$$l_1^{-1} l_2 \in f_{11} G_{11} \sigma_1 G_{21} f_{21}^{-1} \cap f_{12} G_{12} \sigma_2 G_{22} f_{22}^{-1} \cap \ldots$$
$$\cap f_{1n} G_{1n} \sigma_n G_{2n} f_{2n}^{-1} \quad (3)$$

   where G_{ij} is the symmetry group of primitive feature j of S_i and f_{ij} is its feature coordinates.

2. m solids have a chaining general contact, the relative location of solid m with respect to solid 1:

$$l_1^{-1} l_m \in f_1 G_{12} \sigma_1 G_{21} f_{21}^{-1} f_2 G_{23} \sigma_2 G_{32} f_{32}^{-1} \ldots$$
$$f_{m-1} G_{(m-1)m} \sigma_{m-1} G_{m(m-1)} f_{m(m-1)}^{-1} \quad (4)$$

   where G_{ij} is the symmetry group of the surface on solid i in contact with solid j.

Let us first give a denotation for such a set of contacting surfaces, and then determine what the symmetry group of this collection of surfaces should be.

Definition 3.0.1 *A* **compound feature** $F = (S, \rho)$ *of primitive features* $F_1 = (S_1, \rho_1), ..., F_n = (S_n, \rho_n)$, *is defined to be*

- $S = S_1 \cup ... \cup S_n$
- $\rho = \rho_1 \cup ... \cup \rho_n$

The advantage of using a relation ρ to denote the orientations of a feature (Definition 2.1.2) becomes more obvious for compound features. When two primitive features are combined, there often are multiple normal directions at the points where the surfaces meet (Figure 3).

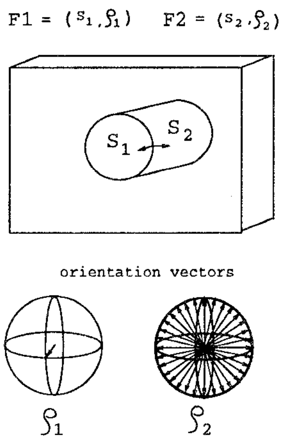

Figure 3: A pair of **distinct** features F_1, F_2

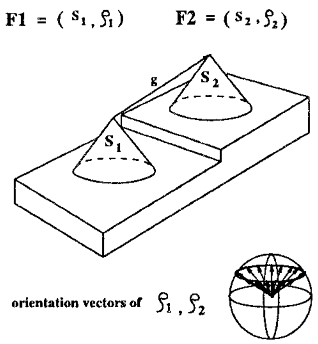

Figure 4: Two conic features F_1, F_2 which are **1-congruent** to each other

3.1 Pairwise Relationship of Oriented Features

In order to determine the symmetry group of a compound feature systematically, we start with the simplest compound feature — a compound feature composed of only one pair of primitive features. See Figures 3, 4 and 5 for examples of these simple compound features (Note that only a finite face on each primitive feature is drawn).

Given a pair of primitive features, what kind of relationship holds between the two features and what is

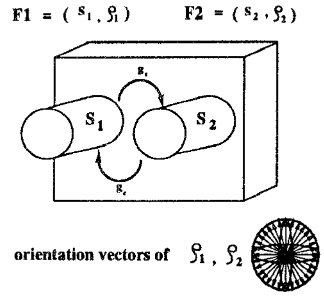

Figure 5: Two cylindrical features F_1, F_2 which are **2-congruent** to each other

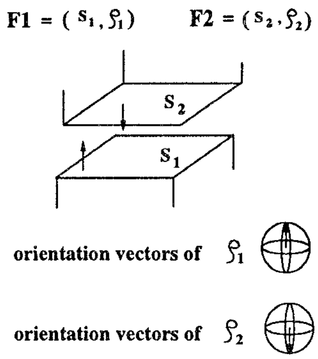

Figure 6: Two complementary features F_1, F_2

the effect of such a relationship in terms of determining their symmetry group? The following definition gives a characterization of four relationships between a pair of primitive features:

Definition 3.1.2 *Two oriented primitive features* $F_1 = (S_1, \rho_1), F_2 = (S_2, \rho_2)$ *are said to be*

- **Distinct:** if for any open subsets $S_1' \subset S_1, S_2' \subset S_2$, no $g = tr \in \mathcal{E}^+$ exists such that $g(S_1') \subset S_2$ or $g(S_2') \subset S_1$. See Figure 3 for an example of a pair of distinct features F_1, F_2.
- **1-congruent**: if there exists at least one $g \in \mathcal{E}^+$ such that $g(S_1) = S_2$ and $g * \rho_1 = \rho_2$, but for all such $g, g(S_2) \neq S_1$. For an example see Figure 4. Another example is two parallel planar surfaces with normal vectors pointing in the same direction.
- **2-congruent**: if there exists $g_c \in \mathcal{E}^+$ such that $g_c(S_1) = S_2, g_c(S_2) = S_1, g_c * \rho_1 = \rho_2$ and $g_c * \rho_2 = \rho_1$. For an example, consider two parallel cylindrical surfaces having the same radius and normal vectors pointing away from their center lines, as in Figure 5. Also, two parallel planar surfaces with normal vectors pointing to the opposite directions serve as examples of a pair of 2-congruent features.
- **Complementary:** if there exists $g \in \mathcal{E}^+$ such that $g(S_1) = S_2$ and $g * \rho_1 = -\rho_2$ where $-\rho_2 = \{(s, -v) | (s, v) \in \rho_2\}$; in other words, $\forall (s, v) \in$

$g * \rho_1, \exists(s, -v) \in \rho_2$, and $\forall(s, v) \in \rho_2, \exists(s, -v) \in g * \rho_1$. See Figure 6 for an example.

It is easy to verify that these relationships are symmetrical and exhaustive.

Proposition 3.1.3 Distinct, 1-congruent, 2-congruent *and* complementary *are the only possible relationships between a pair of primitive features.*

Corollary 3.1.4 *Except for a pair of planar surface primitive features,* distinct, 1-congruent, 2-congruent *and* complementary *relationships are mutually exclusive relations between a pair of primitive features.*

The definition for oriented features allows us to distinguish a feature from its complement which we cannot do for features treated only as sets. In general the relationship between two primitive features can be either distinct, 1-congruent, 2-congruent or complementary, except for a pair of planar surfaces of solids which are always complementary of each other and at the same time can be either 1-congruent or 2-congruent.

When two solids have a surface contact, it is the case that two features which are complementary of each other are brought into coincidence. The following proposition states how the symmetry groups of a pair of complementary features are related to each other.

Proposition 3.1.5 *If features* $F_1 = (S_1, \rho_1), F_2 = (S_2, \rho_2)$ *are complementary of each other, where* $a(S_1) = S_2, a \in \mathcal{E}^+$, *and* G_1, G_2 *are the symmetry groups of* F_1, F_2 *respectively, then* $aG_1a^{-1} = G_2$. *In particular, if* $S_1 = S_2$ *then* $G_1 = G_2$ (**the necessary condition for surface contact**).

3.2 Symmetry Group of Multiple Oriented Surfaces

In the next few propositions we shall explore how the symmetry group of a compound feature is expressed by the symmetry groups of its component primitive features.

Proposition 3.2.6 *Given a compound feature* $F = (S, \rho)$ *of primitive features* $F_1 = (S_1, \rho_1), ..., F_n = (S_n, \rho_n)$ *where* $F_1, ..., F_n$ *are pairwise distinct primitive features with symmetry groups* $G_1, ...G_n$ *respectively. Then the symmetry group* G *of* F *is* $G = G_1 \cap ... \cap G_n$.

Proposition 3.2.7 *Let a compound feature* $F = (S, \rho)$ *be composed of a pair of primitive features* $F_1 = (S_1, \rho_1)$ *and* $F_2 = (S_2, \rho_2)$ *which are 1-congruent of each other. If* G_1, G_2 *are the symmetry groups of* F_1, F_2 *respectively, and* G *is the symmetry group of* F *then* $G = G_1 \cap G_2$.

Proposition 3.2.8 *Let a compound feature* $F = (S, \rho)$ *be composed of a pair of primitive features* F_1 *and* F_2 *which are 2-congruent of each other via* g_c *(Definition 3.1.2). If* $F_1 = (S_1, \rho_1), F_2 = (S_2, \rho_2)$ *have symmetry groups* G_1, G_2 *respectively, and* G *is the symmetry group of* F *then* $G = < g_c > (G_1 \cap G_2)$ *where* $< g_c >$ *denotes the subgroup of* $\mathcal{E}^+$ *generated by* g_c.

In general, the symmetry group G of a compound feature F can be found from the intersection of the symmetry groups G_i of its primitive features. When 2-congruent features exist, where the mappings which flips 2-congruent features in F may also contribute to G. These are the new symmetries that do not exist in any individual G_i but only when all the F_is are considered collectively, and the new group they generated is a discrete group.

With this proposition we end this section where propositions are proved for the symmetry groups of all the possible pairs of the oriented primitive features.

4 Applications

As one can observe from the proved results for surface contact, the intersection of symmetry groups of the primitive features is one of the crucial operations in determining the relative motions of contacting solids. We face two computational problems:

1. How to denote symmetry groups, which can be finite, infinite, discrete or continuous, on computers?

2. How to intersect subgroups of $\mathcal{E}^+$ on computers efficiently?

We have successfully implemented an efficient group intersection algorithm using geometric invariants denotation of the groups [5]. The basic symmetry group of each surface of a solid is obtained by a straightforward mapping from the boundary (surface) file of the solid to their respective canonical symmetry groups.

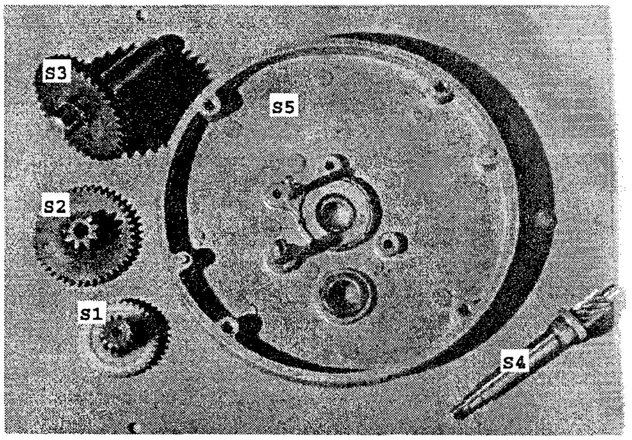

Figure 7: A five-part Gearbox

As an example of assembly specification using symmetry groups, see Figures 7 and 8 for a five-part gearbox. The representation of the assembly is shown in

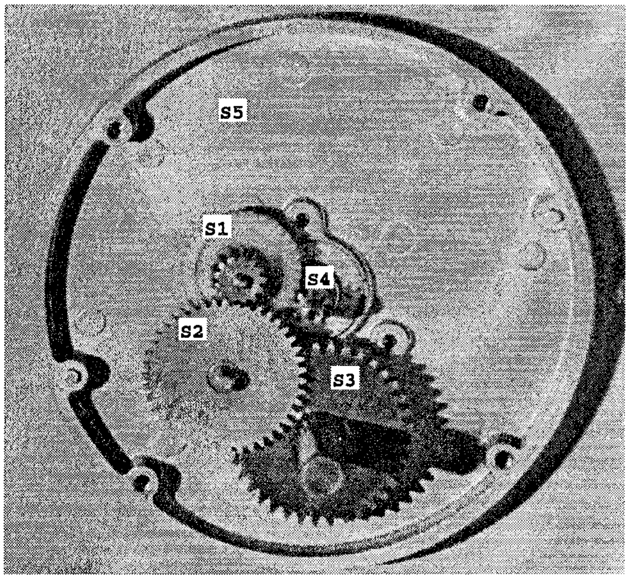

Figure 8: The top view of the gearbox when it is assembled.

Figure 9, where $L_i, i = 1..4$ is the symmetry group of the contacting compound feature between solids S_i and S_5. $L_i = a_i SO(2) a_i^{-1}, i = 1..4$, $SO(2)$ is a one degree rotation group resulted from the intersection of the symmetry group of a plane with that of a cylinder (the compound feature composed of two surfaces of the shaft of a gear). $L_{ij} = L_i L_j = a_{ij} SO(2) b_{ij} SO(2) c_{ij}, i, j = 1..4$ indicate the relative positions between gears (non-surface contact) are simply determined by some translations a_{ij}, b_{ij}, c_{ij}, where the relative gear pitch ratio is also embedded, and a rotation in $SO(2)$. This representation of the gearbox (Figure 9) specifies precisely the articulated gearbox assembly.

Figure 10 from [14] shows a nonlinearizable assembly. Using our representation, one can immediately determine it is a nonlinearizable assembly by computing the symmetry group of the contacting surfaces for each individual part under any possible motion. The result is an identity group, meaning no existing relative motions between the part and the rest of the assembly that can separate the part. A disassemblable subassembly can be identified by computing the symmetry group of a compound feature composed of contacting surfaces from 2 or 3 parts under a disassembly motion, If the resulting group remains as an identity group then the subassembly is not movable. In this example, the resulting group of contacting surfaces for a two-part subassembly, such as $S_1 \cup S_2$, is a T_1 group indicating a one dimensional translation which can separate the assembly.

5 Summary and Discussion

In this paper we have carefully examined the representation and computation aspects for rigid and articulated assembly. Special attention is given to the computational characterization of the symmetry group of multiple contacting surfaces. Here are some questions

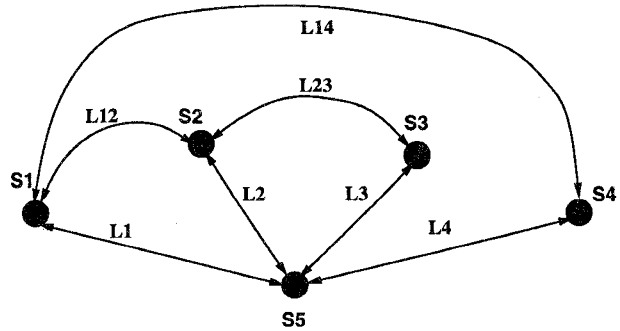

Figure 9: Representation of the gearbox assembly in terms of contacting compound feature symmetry groups, where $L_i = a_i SO(2) a_i^{-1}, i = 1..4$ and $L_{ij} = L_i L_j = a_{ij} SO(2) a_{ij}^{-1}, i, j = 1..4$

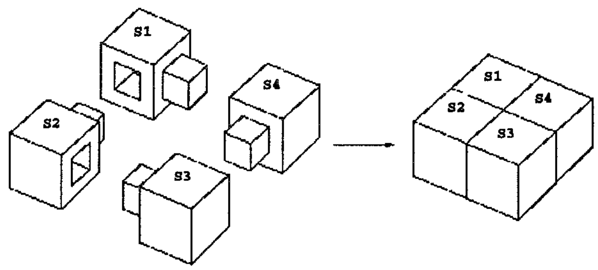

Figure 10: A four-part nonlinearizable assembly from [13].

we seek the answers for:

1. Given two solids S_1, S_2, what is the relative location of the two under n surface contact (n primitive features from each side, Figure 11)?

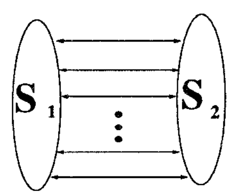

Figure 11: Solids S_1 and S_2 have n contacts

2. Given two solids S_1, S_2, what is the relative location of the two under n general contact?

3. Given m solids in a chaining general contact (Figure 12), what is the relative location of the mth solid with respect to the first solid?

The hypothesis is that using a group theoretical formalization of the oriented surfaces of solids in contact, these questions can be answered automatically and precisely. What we have achieved so far includes a thorough understanding of surface contact among solids, an algorithm to compute the symmetry groups

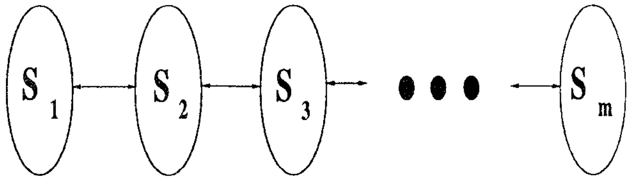

Figure 12: Solids $S_1, S_2, ...S_m$ form a chain

in case (1) above which has been implemented and used in an assembly planner; for cases (2) and (3), we are able to construct the precise expressions for relative positions in terms of contacting surface symmetry groups (equations (3), (4)).

Though the results are promising, many open problems remain. Further work is needed for a computational treatment of *group products* as what have been implemented for group intersections in [4, 5]. The specification of the non-full-range motions existing in an assembly (The first example given in Section 4 contains full range of the rotational motion) is one topic under our current investigation. It is also useful to further study those compound features with more complicated inner structures. For example, one may define a concept of n-congruence on n features $F_1 \dots F_n$ as requiring that there exists $g \in \mathcal{E}^+$ such that $g(F_i) = F_{(i \bmod n)+1}$. Such congruences will give rise to new symmetries of the compound feature. However Proposition 3.2.8 is not trivially generalized to such a proposition:

Proposition 5.0.9 *Given a compound feature* $F = (S, \rho)$ *of primitive features* $F_1 = (S_1, \rho_1), ..., F_n = (S_n, \rho_n)$ *with symmetry groups* $G_1, ...G_n$ *respectively, the symmetry group* G *of* F *is*

$$G = < \{g_{ij}\} > (G_1 \cap ... \cap G_n)$$

where $\{g_{ij}\}$ *is a set of isometries* $g_{ij} \in \mathcal{E}^+$*, each of which is associated with a pair of 2-congruent primitive features* F_i, F_j *in* F *such that* F_i, F_j *are 2-congruent via* g_{ij}*;* $< \{g_{ij}\} >$ *denotes the group generated by all such* g_{ij}*s.*

In summary, our new results on oriented surfaces has laid out a realistic and precise group theoretic framework for characterizing surfaces of solids and capture the very nature of surface contact — the state of being complementary. Under this formalization surface contact can be treated conceptually effectively and computationally efficiently. In this paper we have generalized this framework and applied it to provide a concise, complete and computational representation for rigid and articulated assembly. Though the approach is theoretical, the results are easily implemeted for various applications involving solids in contact, provided the input solid can be expressed in terms of its algebraic surfaces (a boundary file) as many current solid modeling systems are able to do.

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